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The Power Series Ring over an Ore Domain Need Not Be Ore

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A ring R satisfies the right Ore condition iff for all nonzero divisors, r, $s \in R$, $rR \cap sR \neq 0$. If, in addition to satisfying the right Ore condition, R has no zero divisors, then R is said to be a right Ore domain. Ore's famous theorem states that R satisfies the right Ore condition iff R has a right quotient ring.

It has long been an open question whether the property of being a right Ore domain lifts to the power series ring. This is known to hold if the right Ore domain is right Noetherian or if it satisfies a polynomial identity. In this paper we construct a right Ore domain R such that its power series ring $R[\![t]\!]$ fails to satisfy the right Ore condition.

It is well known that all polynomial rings and matrix rings over a right Ore domain satisfy the right Ore condition (see [3, p. 439]). To gain insight into the construction of our example, we next review the proof for the polynomial ring $R[t]$ over a right Ore domain R. Let Q be the quotient division ring of R. Using the division algorithm we see that $Q[t]$ is a right Ore domain. Let $q(t)$ be an arbitrary polynomial in $Q[t]$. Because R satisfies the right Ore condition, the coefficients of $q(t)$ have a common denominator. Hence $q(t)$ has the form $r(t)\,s^{-1}$, where $r(t) \in R[t]$, $s \in R$. Let $f(t)$, $g(t)$ be polynomials in $R[t]$. Then, because $Q[t]$ is a right Ore domain, there exist $q_1(t) = r_1(t)\,s_1^{-1}$, $q_2(t) = r_2(t)\,s_2^{-1}$ in $Q[t]$ such that $f(t)\,r_1(t)\,s_1^{-1} = g(t)\,r_2(t)\,s_2^{-1}$. Using that R satisfies the right Ore condition, we immediately have $f(t)\,R[t] \cap g(t)\,R[t] \neq 0$. Hence, $R[t]$ is a right Ore domain. A similar approach yields a proof for the satisfaction of the right Ore condition by the matrix rings over a right Ore domain. Heavily used in the preceding argument is the finite number of coefficients in $q(t)$. In a power series ring we lose this finiteness. Our example exploits that fact.

Of crucial importance for the construction of our example is Cohn's theorem which states that every fir is embeddable in a division ring [2]. Recall that the free product of any family of division rings over a common subdivision ring is a fir [1]. We construct the power series ring example by

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taking a direct limit of subrings of such free products. Next we define these subrings.

Let D_0 be an arbitrary division ring with center F. Let R_0 be a subring of D_0 containing F and let $X_0 = \{x_0, y_0, z_0\}$ be a set of commuting indeterminants over F. Consider the free product $D_0 *_F F(X_0)$, where $F(X_0)$ is the quotient field of the polynomial ring $F[X_0] = F[x_0, y_0, z_0]$. By Cohn's theorem, the ring $D_0 *_F F(X_0)$ is embeddable in a division ring D_1. We define R_1 to be the subring of D_1 generated by R_0 and $\{x_0, y_0, r^{-1}z_0$, for all nonzero r in $R_0\}$. That is, $R_1 = R_0\langle x_0, y_0, r^{-1}z_0$, for all $r \neq 0, r \in R_0\rangle$.

Assume we have defined the ring R_i such that R_i contains F and R_i is contained in a division ring D_i. Let $X_i = \{x_i, y_i, z_i\}$ be a set of commuting indeterminates over F. Again we consider the free product $D *_F F(X_i)$. Then $R_{i+1} = R_i\langle x_i, y_i, r^{-1}z_i$, for all $r \neq 0, r \in R_i\rangle$.

Let $R = \bigcup R_i$. The construction of R quickly yields R to be a right Ore domain. For let $r, s \in R$. Then there exists n such that $r, s \in R_n$. Note that $r^{-1}z_n, s^{-1}z_n \in R$; hence $z_n \in rR \cap sR$. Therefore, R satisfies the right Ore condition.

LEMMA. *For all $m > n$ there exists a canonical R_n-bimodule homomorphism $\pi: R_m \to R_n$ which extends the identity on R_n. Hence $R = R_n \oplus \ker \pi$ as R_n-bimodules.*

Proof. It suffices to prove the lemma for $m = n + 1$. Consider R_{n+1} as a subring of the free algebra $D_n\langle X_n\rangle$. We define a grading on $D_n\langle X_n\rangle$ by letting elements in D_n have degree 0 and elements in $\{x_n, y_n, z_n\}$ have degree 1. By restricting this grading to R_{n+1} we see that R_{n+1} is a graded ring with R_n as the degree zero part. Hence, the projection map from R_{n+1} to its zeroth degree component R_n is the desired R_n-bimodule map.

To show that the power series ring $R[\![t]\!]$ does not satisfy the right Ore condition, we need only produce two nonzero elements $x, y \in R[\![t]\!]$ such that $xR[\![t]\!] \cap yR[\![t]\!] = 0$. We claim that $x = \sum_{i=0}^{\infty} x_i t^i$, $y = \sum_{i=0}^{\infty} y_i t^i$ are such elements. Suppose instead that $xR[\![t]\!] \cap yR[\![t]\!] \neq 0$. Then there exist $u = \sum_{i=0}^{\infty} u_i t^i$, $v = \sum_{i=0}^{\infty} v_i t^i$, $u_i, v_i \in R$, such that $xu = yv$. Without loss of generality we may assume $u_0 \neq 0 \neq v_0$. Because $u_0 \in R$, there exists n such that $u_0 \in R_n$. Considering the coefficients of t^n in $xu = yv$ we obtain

$$x_n u_0 + \sum_{i=0}^{n-1} x_i u_{n-i} = y_n v_0 + \sum_{i=0}^{n-1} y_i v_{n-i}. \tag{1}$$

All terms in (1) occur in R_m, for some $m > n$. Let $\pi_1: R_m \to R_{n+1}$ be the canonical homomorphism given by the lemma. Applying π_1 to (1) we have

$$x_n u_0 = y_n \bar{v}_n + \sum_{i=0}^{n-1} (x_i \bar{u}_i - y_i \bar{v}_i), \tag{2}$$

where $\bar{u}_i = \pi_1(u_{n-i})$, $\bar{v}_i = \pi_1(v_{n-i})$. All terms in Eq. (2) are contained in $R_{n+1} \subset D_n\langle x_n, y_n, z_n\rangle$. Evaluating (2) for $y_n = 0 = z_n$ we see that

$$1x_n u_0 = \sum_{i=0}^{n-1} (x_i \hat{u}_i - y_i \hat{v}_i), \tag{3}$$

where $\hat{u}_i, \hat{v}_i \in D_n\langle x_n\rangle$. Because $D_n\langle x_n\rangle$ is a graded ring, we may assume $\hat{u}_i$, $\hat{v}_i$ to be linear in x_n. That is, $\hat{u}_i = \sum_j \alpha_{ij} x_n \beta_{ij}$, $\hat{v}_i = \sum_j \gamma_{ij} x_n \delta_{ij}$, for some α_{ij}, β_{ij}, γ_{ij}, $\delta_{ij} \in R_n$. Note that $R_n x_n R_n$ is isomorphic to $R_n \otimes_F R_n$. Hence (3) implies

$$1 \otimes u_0 = \sum_{i,j} ((x_i \alpha_{ij} \otimes \beta_{ij}) - (v_i \gamma_{ij} \otimes \delta_{ij})). \tag{4}$$

The canonical map $\pi_2\colon R_n \to R_0$ given by the lemma induces a map $\pi_2 \otimes 1$: $R_n \otimes R_n \to R_0 \otimes_F R_n$. Applying $\pi_2 \otimes 1$ to (4) we have $1 \otimes u_0 = 0$, which forces u_0 to be zero because the tensor product is over F, a field. But u_0 was assumed to be nonzero. Thus, $xR[\![t]\!] \cap yR[\![t]\!] = 0$ and, so, $R[\![t]\!]$ is not an Ore domain.

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